Boundedness for commutators of some maximal functions on the p-adic vector space

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Abstract: In this paper, the main aim is to demonstrate the boundedness for commutators of sharp maximal function in the context of the p-adic variable Lebesgue spaces and Morrey spaces. where the symbols of the commutators belong to the p-adic version of Lipschitz or BMO spaces. Moreover, the boundedness of commutators of fractional maximal operator in the p-adic Morrey space is also given, where the symbols of the commutators belong to the p-adic version of Lipschitz space, by which some new characterizations of Lipschitz and BMO spaces are obtained.

Keywords: p-adic field; sharp maximal function; fractional maximal function; commutator; variable Lebesgue space; Lipschitz space; BMO space

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1 Introduction and main results

For a prime number p. The p-adic field is consist of $\mathbb{Q}$ with respect to non-Archimedean p-adic norm. Let $x = p^{\gamma}\frac{a}{b}$, where $x \in \mathbb{Q}$ and $\gamma \in \mathbb{Z}$, a and b are integers which are not divisible by p, then the p-adic norm is defined by $|x|_p = p^{-\gamma}$ and satisfies following properties.

(i) $|x|_p \geq 0$. Specially, $|x|_p = 0$ if and only if $x = 0$;

(ii) $|xy|_p = |x|_p|y|_p$;

(iii) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. If $|x|_p \neq |y|_p$, then the equality holds.

From the standard p-adic analysis, p-adic number x $(x \neq 0)$ can be written as

$$x = p^{\gamma}(a_0 + a_1 p + a_2 p^2 + \cdots) = p^{\gamma}\sum_{j=0}^{\infty} a_j p^j, \quad a_j = 0, \cdots, p-1, \ a_0 \neq 0.$$

Naturally, the aforementioned p-adic number x converges.

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For any $x = (x_1, x_2, \cdots, x_n)$, where $x_i \in \mathbb{Q}_p$ $(i = 1, ..., n)$, the p-adic norm is defined by $|x|_p = \max_{1\le j\le n} |x_j|_p$. Moreover, the p-adic ball is denoted by $B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p \le p^\gamma\}$, where the center of p-adic ball $a \in \mathbb{Q}_p^n$ and radius p^γ with $\gamma \in \mathbb{Z}$. The p-adic sphere is written as $S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\} = B_\gamma(a) \setminus B_{\gamma-1}(a)$. It is easy that $B_\gamma(a) = \bigcup_{k\le\gamma} S_k(a)$.

In view of $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists Haar measure on $\mathbb{Q}_p^n$, it is easy to know that unique Haar measure dx on $\mathbb{Q}_p^n$ (up to positive constant multiple) satisfies translation invariant. Normalizing the measure dx by $\int_{B_0(0)} dx = |B_0(0)|_h = 1$, where $|B_0(0)|_h$ is denoted by the Haar measure of p-adic unit ball. For any $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$, by using the normalized Haar measure $\int_{B_\gamma(a)} dx = |B_\gamma(a)|_h = p^{n\gamma}$ and $\int_{S_\gamma(a)} dx = |S_\gamma(a)|_h = p^{n\gamma}(1 - p^{-n}) = |B_\gamma(a)|_h - |B_{\gamma-1}(a)|_h$. For more details about the p-adic analysis, see [17, 18].

The study of p-adic harmonic analysis holds significant research importance and occupies a pivotal position in the field of mathematics. It plays an indispensable role in enhancing our understanding and comprehension of various branches such as number theory, algebraic geometry, and representation theory [10, 12, 16]. Additionally, p-adic harmonic analysis also exhibits broad prospects for practical applications, possessing potential value in areas like cryptography, signal processing, and data analysis [5, 15].

Let T be the classical singular integral operator. The Coifman-Rochberg-Weiss type commutator $[b, T]$ generated by T and a suitable function b is defined by

$$[b, T]f = bT(f) - T(bf). \tag{1.1}$$

In [4, 9], (1.1) is bounded on $L^s(\mathbb{R}^n)$ $(1 < s < \infty)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Janson [9] also established some characterizations of the Lipschitz function space $\Lambda_\beta(\mathbb{R}^n)$ via commutator (1.1) and proved that (1.1) is bounded from $L^s(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < s < \frac{n}{\beta}$ and $\frac{1}{s} - \frac{1}{q} = \frac{\beta}{n}$ $(0 < \beta < 1)$ if and only if $b \in \Lambda_\beta(\mathbb{R}^n)$ (see also Paluszyński [14]).

Let $0 \le \alpha < n$, for $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$, the p-adic fractional maximal function of f is defined by

$$M_{\alpha,p}(f)(x) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}} \int_{B_\gamma(x)} |f(y)|dy,$$

where the supremum is taken over all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$. For $\alpha = 0$, we write $M_p = M_{0,p}$. If $b \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$, the fractional maximal commutator produced by b with $M_{\alpha,p}$ is provided by

$$M^b_{\alpha,p}(f)(x) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(x) - b(y)||f(y)|dy. \tag{1.2}$$

For $\alpha = 0$, we write $M^b_p = M^b_{0,p}$. And the commutator produced by b with $M_{\alpha,p}$ is defined by

$$[b, M_{\alpha,p}](f)(x) = b(x)M_{\alpha,p}(f)(x) - M_{\alpha,p}(bf)(x). \tag{1.3}$$

For $\alpha = 0$, we write $[b, M_p] = [b, M_{0,p}]$.

Kim [11] introduced the p-adic sharp maximal function.

$$M_p^{\sharp}(f)(x) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}| dy,$$

where $f_{B_\gamma(x)}$ is the average of f over $B_\gamma(x)$. And the nonlinear commutators produced by b with $M_p^{\sharp}$ is defined by

$$[b, M_p^{\sharp}](f)(x) = b(x)M_p^{\sharp}(f)(x) - M_p^{\sharp}(bf)(x). \tag{1.4}$$

It is worth noting that operators (1.2) and (1.3) essentially differ from each other. Indeed, (1.2) is positive and sublinear, but (1.3) is neither positive nor sublinear. So does (1.4).

In the Euclidean setting, there are many people studied the operators (1.2)-(1.4), see [1, 22–28] for instance, we can study some results on p-adic fields by borrowing similar methods.

Recently, when b belongs to Lipschitz spaces, the authors [20, 21] gave the necessary and sufficient conditions for the boundedness of the commutator (1.2)-(1.4) in the variable Lebesgue space and Morrey space. When b belongs to BMO spaces, the similar result can be obtained in [19].

Inspired by the above literature [19–21], we focus on studying the boundedness for commutators of sharp maximal function in the p-adic variable Lebesgue spaces and Morrey spaces. Moreover, the boundedness of commutators of fractional maximal operator in the p-adic Morrey space is also given, by which some new characterizations of Lipschitz and BMO spaces are obtained.

In order to introduce the following theorem, the b^- is defined by

$$b^-(x) = \begin{cases} 0, & \text{if } b(x) \geq 0, \\ |b(x)|, & \text{if } b(x) < 0, \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$. Then the following result give boundedness for commutator of sharp maximal function on p-adic variable Lebesgue space. Moreover, new characterizations of BMO spaces are obtained.

Theorem 1.1 Assume that $b \in L^1_{loc}(\mathbb{Q}_p^n)$, then the following statements are equivalent.

(1) $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$ and $b^- \in L^\infty(\mathbb{Q}_p^n)$;

(2) $[b, M_p^{\sharp}]$ is bounded on $L^{q(\cdot)}(\mathbb{Q}_p^n)$; for all $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$,

(3) $[b, M_p^{\sharp}]$ is bounded on $L^{q(\cdot)}(\mathbb{Q}_p^n)$; for some $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$,

(4) There exists some $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$,

$$\sup_{\substack{\gamma\in\mathbb{Z} \\ x\in\mathbb{Q}_p^n}} \frac{\|(b - \frac{p^2}{2(p-1)} M_p^{\sharp}(b\chi_{B_\gamma(x)})\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} < \infty \tag{1.5}$$

(5) For all $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, such that (1.5) holds.

Remark 1 If $q(\cdot)$ be a constant exponent, this result can be obtained in Theorem 1.4 of [7].

The following result give boundedness for commutator of sharp maximal function on p-adic Morrey space. Moreover, new characterizations of Lipschitz spaces are obtained.

Theorem 1.2 Assume that $b \in L^1_{loc}(\mathbb{Q}_p^n), 0 < \beta < 1$, then the following statements are equivalent.

(1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$;

(2) $[b, M_p^\sharp]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$; for all r, q with $1 < r < n/\beta$ and $1/q = 1/r - \beta/(n-\lambda)$

(3) $[b, M_p^\sharp]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$; for some r, q with $1 < r < n/\beta$ and $1/q = 1/r - \beta/(n-\lambda)$

(4) There exists some r, q with $1 < r < n/\beta$ and $1/q = 1/r - \beta/(n-\lambda)$, such that

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \frac{\|(b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} < \infty \tag{1.6}$$

(5) For all r, q with $1 < r < n/\beta$ and $1/q = 1/r - \beta/(n-\lambda)$, such that (1.6) holds

Remark 2 In the Euclidean setting, we can see Theorem 1.3 of [6].

The following result give boundedness for commutator of sharp maximal function on p-adic Morrey space. Moreover, new characterizations of BMO spaces are obtained.

Theorem 1.3 Assume that $b \in L^1_{loc}(\mathbb{Q}_p^n)$, then the following statements are equivalent.

(1) $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$ and $b^- \in L^\infty(\mathbb{Q}_p^n)$;

(2) $[b, M_p^\sharp]$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$, for all q with $1 < q < \infty$

(3) $[b, M_p^\sharp]$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$, for some q with $1 < q < \infty$

(4) There exists some q with $1 < q < \infty$, such that

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{\|(b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} < \infty \tag{1.7}$$

(5) For all q with $1 < q < \infty$ such that (1.7) holds

Theorem 1.4 (**Spanne type result**) Assume that b be a locally integral function on $\mathbb{Q}_p^n$, and $0 \leq \alpha < \alpha + \beta < n$. Let $1 < r < n/(\alpha+\beta)$, $0 < \lambda < n - r(\alpha+\beta)$, for $1/q = 1/r - (\alpha+\beta)/n$, and $\lambda/r = \kappa/q$ then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ if and only if $M^b_{\alpha,p}$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\kappa}(\mathbb{Q}_p^n)$.

Remark 3 When $\alpha = 0$, the above result can be found in Theorem 3 of [8].

Theorem 1.5 (**Adams type result**) Assume that b be a locally integral function on $\mathbb{Q}_p^n$, and $0 < \alpha < \alpha + \beta < n$. Let $1 < r < n/(\alpha + \beta)$, $0 < \lambda < n - r(\alpha + \beta)$, for $1/q = 1/r - (\alpha + \beta)/(n - \lambda)$ then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ if and only if $M_{\alpha,p}^b$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$.

Remark 4 For the case of $\alpha = 0$, the above result can be obtained in Theorem 2 of [8].

Theorem 1.6 (**Spanne type result**) Assume that b be a locally integral function on $\mathbb{Q}_p^n$, and $0 \leq \alpha < \alpha + \beta < n$. Let $1 < r < n/(\alpha + \beta)$, $0 < \lambda < n - r(\alpha + \beta)$, for $1/q = 1/r - (\alpha + \beta)/n$, and $\lambda/r = \kappa/q$ then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$ if and only if $[b, M_{\alpha,p}]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\kappa}(\mathbb{Q}_p^n)$.

Remark 5 For the case of $\alpha = 0$, the above result can be obtained in Theorem 7 of [8].

Theorem 1.7 (**Adams type result**) Assume that b be a locally integral function on $\mathbb{Q}_p^n$, and $0 < \alpha < \alpha + \beta < n$. Let $1 < r < n/(\alpha + \beta)$, $0 < \lambda < n - r(\alpha + \beta)$, for $1/q = 1/r - (\alpha + \beta)/(n - \lambda)$ then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$ if and only if $[b, M_{\alpha,p}]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$.

Remark 6 For the case of $\alpha = 0$, the above result can be obtained in Theorem 6 of [8].

Throughout this paper, the letter C always takes place of a constant independent of the primary parameters involved and whose value may differ from line to line. In addition, we give some notations. Here and hereafter $|E|_h$ will always denote the Haar measure of a measurable set E on $\mathbb{Q}_p^n$ and by χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{Q}_p^n$.

2 Preliminaries

2.1 p-adic function spaces

Let $1 \leq q < \infty$, a measurable function f is said to belong to the p-adic Lebesgue space $L^q(\mathbb{Q}_p^n)$ if

$$\|f\|_{L^q(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(x)|^q dx\right)^{\frac{1}{q}} < \infty.$$

Moreover, for $q = \infty$ and denote $L^\infty(\mathbb{Q}_p^n)$ as the set of all measurable real-valued functions f on $\mathbb{Q}_p^n$ satisfying

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)} = \operatorname*{ess\,sup}_{x \in \mathbb{Q}_p^n} |f(x)| = \inf\left\{\lambda > 0 : \left|\{x \in \mathbb{Q}_p^n : |f(x)| > \lambda\}\right|_h = 0\right\} < \infty.$$

Here, if the limit exists, the integral in above equation is defined as follows:

$$\int_{\mathbb{Q}_p^n} |f(x)|^q dx = \lim_{\gamma\to\infty} \int_{B_\gamma(0)} |f(x)|^q dx = \lim_{\gamma\to\infty} \sum_{-\infty<k\leq\gamma} \int_{S_k(0)} |f(x)|^q dx.$$

Given that a measurable function $q(\cdot)$ is a variable exponent if $q(\cdot):\mathbb{Q}_p^n \to (0.\infty)$. In [2], the following definition is introduced.

Definition 2.1 Given a measurable function $q(\cdot)$ defined on $\mathbb{Q}_p^n$, we denote by

$$q_- := \operatorname*{ess\,inf}_{x\in\mathbb{Q}_p^n} q(x), \quad q_+ := \operatorname*{ess\,sup}_{x\in\mathbb{Q}_p^n} q(x).$$

(1) $q'_- = \operatorname{ess\,inf}_{x\in\mathbb{Q}_p^n} q'(x) = \frac{q_+}{q_+-1}$, $q'_+ = \operatorname{ess\,inf}_{x\in\mathbb{Q}_p^n} q'(x) = \frac{q_-}{q_--1}$
(2) Denote by $\mathscr{P}(\mathbb{Q}_p^n)$ the set of all measurable function $q(\cdot):\mathbb{Q}_p^n \to (1,\infty)$ such that

$$1 < q_- \leq q(x) \leq q_+ < \infty, \quad x \in \mathbb{Q}_p^n.$$

Definition 2.2 (**p-adic variable exponent Lebesgue spaces**) Let $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$. Define the p-adic variable exponent Lebesgue spaces $L^{q(\cdot)}(\mathbb{Q}_p^n)$ as follows

$$L^{q(\cdot)}(\mathbb{Q}_p^n) = \{f \text{ is measurable function} : \mathscr{F}_q(\ \frac{f}{\eta}\ ) < \infty \text{ for some constant } \eta > 0\},$$

where $\mathscr{F}_q(f) := \int_{\mathbb{Q}_p^n} |f(x)|^q(x)dx$. The Lebesgue space $L^{q(\cdot)}(\mathbb{Q}_p^n)$ is a Banach function space with respect to the Luxemburg norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} = \inf\{\eta > 0 : \mathscr{F}_q(\ \frac{f}{\eta}\ ) = \int_{\mathbb{Q}_p^n} \left(\frac{|f(x)|}{\eta}\right)^{q(x)} dx \leq 1\}.$$

Definition 2.3 (log $-$Hölder **continuity**) Let measurable function $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$.
(1) Denote by $\mathscr{C}_0^{\log}(\mathbb{Q}_p^n)$ the set of all $q(\cdot)$ which satisfies

$$\gamma(q_-(B_\gamma(x)) - q_+(B_\gamma(x))) \leq C$$

for all $\gamma \in \mathbb{Z}$ and any $x \in \mathbb{Q}_p^n$, where C denotes a universal constant.
(2) The set $\mathscr{C}_\infty^{\log}(\mathbb{Q}_p^n)$ consists of all $q(\cdot)$ which satisfies

$$|q(x) - q(y)| \leq \frac{C}{\log_p(p + \min\{|x|_p, |y|_p\})}$$

for any $x, y \in \mathbb{Q}_p^n$, where C denotes a universal constant.
(3)(see [19]) Denote by $\mathscr{C}^{\log}(\mathbb{Q}_p^n) = \mathscr{C}_0^{\log}(\mathbb{Q}_p^n) \cap \mathscr{C}_\infty^{\log}(\mathbb{Q}_p^n)$ the set of all global log $-$Hölder continuous functions $q(\cdot)$.

Kim[11] gave the following definition of the p-adic version of BMO space.

Definition 2.4 Let $f \in L^1_{loc}(\mathbb{Q}_p^n)$ be given. If $\|M_p^\sharp(f)\|_{L^\infty(\mathbb{Q}_p^n)} < \infty$, then we say that f is a function of bounded mean oscillation on $\mathbb{Q}_p^n$). We denote the space of such function by $\mathrm{BMO}(\mathbb{Q}_p^n)$; that is to say,

$$\mathrm{BMO}(\mathbb{Q}_p^n) = \{f \in L^1_{loc}(\mathbb{Q}_p^n) : M_p^\sharp(f) \in L^\infty(\mathbb{Q}_p^n)\}.$$

For $f \in \mathrm{BMO}(\mathbb{Q}_p^n)$, we write

$$\|f\|_{\mathrm{BMO}(\mathbb{Q}_p^n)} = \|M_p^\sharp(f)\|_{L^\infty(\mathbb{Q}_p^n)} = \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}| dy,$$

where $f_{B_\gamma(x)}$ is the average of f over $B_\gamma(x)$.

The following result introduce the basic definition of p-adic Lipschitz spaces [3].

Definition 2.5 Let $0 < \beta < 1$, the p-adic version of homogeneous Lipschitz spaces $\Lambda_\beta(\mathbb{Q}_p^n)$ is defined by

$$\Lambda_\beta(\mathbb{Q}_p^n) := \{f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{Q}_p^n) : \ \|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} = \sup_{x,y\in\mathbb{Q}_p^n,\ x\neq y} \frac{|f(x) - f(y)|}{|x-y|_p^\beta}.$$

Remark 7 (1) Assume that $1 \le q < \infty$, the p-adic version of homogeneous Lipschitz spaces $Lip^q_\beta(\mathbb{Q}_p^n)$ is defined by

$$Lip^q_\beta(\mathbb{Q}_p^n) := \{f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{Q}_p^n) : \ \|f\|_{Lip^q_\beta(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{Lip^q_\beta(\mathbb{Q}_p^n)} = \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|^q dy \right)^{\frac{1}{q}}.$$

(2)(see Lemma 6 of [8])By virtue of Definition 2.5, for all $0 < \beta < 1$ and $1 \le q < \infty$, $\Lambda_\beta(\mathbb{Q}_p^n) \approx Lip^q_\beta(\mathbb{Q}_p^n)$ with equivalent norms.

It is well known that the classic Morrey space were given by Morrey in [13] to study the certain problem to second-order elliptic partial differential equations (PDE).

Definition 2.6 (**Classic Morrey space**) The p-adic version of Morrey space is defined by $L^{q,\lambda}(\mathbb{Q}_p^n)$ as follows, for $1 \le q \le \infty$ and $0 \le \lambda \le n$, if $f \in L^q_{\mathrm{loc}}(\mathbb{Q}_p^n)$ with the finite norm

$$\|f\|_{L^{q,\lambda}} = \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} |B_\gamma(x)|^{-\frac{\lambda}{qn}} \|f\|_{L^q(\mathbb{Q}_p^n)} < \infty.$$

2.2 Auxiliary propositions and lemmas

In this part we state some auxiliary propositions and lemmas which will be needed for proving our main theorems. And we only describe partial results we need.

Firstly, the p-adic version of Hölder's inequality can be obtained in [2].

Lemma 2.1 (**Generalized Hölder's inequality on $\mathbb{Q}_p^n$**) Let $\mathbb{Q}_p^n$ be an n-dimensional p-adic vector space. Suppose that $q_1(\cdot), q_2(\cdot), r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$ and $r(\cdot)$ satisfy $\frac{1}{r(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$ almost everywhere. Then there exists a positive constant C such that for all $f \in L^{q_1(\cdot)}(\mathbb{Q}_p^n)$ and $g \in L^{q_2(\cdot)}(\mathbb{Q}_p^n)$, the inequality holds.

$$\|fg\|_{L^{r(\cdot)}(\mathbb{Q}_p^n)} \leq C\|f\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\|g\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)},$$

The authors in [21] obtained the following Lemmas 2.2, 2.3.

Lemma 2.2 Let $0 < \beta < 1,\ 0 < \alpha < \alpha + \beta < n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, then for any $x \in \mathbb{Q}_p^n$, we have

$$M_{\alpha,p}^b \leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} M_{\alpha+\beta,p}(f)(x).$$

Lemma 2.3 Let $0 < \alpha < n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$, then for any $x \in \mathbb{Q}_p^n$ such that $M_{\alpha,p}(f)(x) < \infty$, we obtain

$$|[b, M_{\alpha,p}](f)(x)| \leq M_{\alpha,p}^b(f)(x).$$

The following result derices from [8].

Lemma 2.4 Assume $0 < \alpha < n,\ 1 < r < n/\alpha$ and $0 < \lambda < n - r\alpha$.
(1) If $1/q = 1/r - \alpha/(n-\lambda)$, then $M_{\alpha,p}$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$.
(2) If $1/q = 1/r - \alpha/n,\ \lambda/r = \kappa/q$, then $M_{\alpha,p}$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\kappa}(\mathbb{Q}_p^n)$.

The following result can be founded in [21].

Lemma 2.5 Let $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, there exists a positive constant C, such that for any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, the equality holds.

$$\frac{1}{|B_\gamma(x)|_h}\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_\gamma(x)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)} \leq C$$

He and Li [8] gave the norm of characteristic function

Lemma 2.6 Let $1 \leq q < \infty$ and $0 < \lambda < n$, then

$$\left\|\chi_{B_\gamma(x)}\right\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} = p^{\frac{n-\lambda}{nq}}$$

3 Proof of the principal results

Proof of Theorem 1.1 *Since the implications* $(2) \Longrightarrow (3)$ *and* $(5) \Longrightarrow (4)$ *follow readily, and* $(2) \Longrightarrow (5)$ *is similar to* $(3) \Longrightarrow (4)$*, we only need to prove* $(1) \Longrightarrow (2), (3) \Longrightarrow (4), (4) \Longrightarrow (1)$.

$(1) \Longrightarrow (2)$*:for any p-adic ball* $B_\gamma(x) \subset \mathbb{Q}_p^n$*, using (3.2) of [20] and* $M_p^\sharp(f)(x) \leq 2M_p(f)(x)$*, we obtain*

$$|[b, M_p^\sharp](f)(x)| \leq 4((b^-(x))M_p(f)(x) + M_p(b^- f)(x)) + 2M_p^{|b|} f(x). \tag{3.1}$$

Thus

$$\|[b, M_p^\sharp](f)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq C(\|b^-\|_{L^\infty(\mathbb{Q}_p^n)} + \|b\|_{\mathrm{BMO}(\mathbb{Q}_p^n)})\|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.$$

Note that $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$*, then* $|b| \in \mathrm{BMO}(\mathbb{Q}_p^n)$*. For* $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ *with* $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$*, using (1), Minkowski's inequality and boundedness of* M_p *and* $M_p^{|b|}$ *on* $L^{q(\cdot)}(\mathbb{Q}_p^n)$ *[2, 19] , we give that* $[b, M_p^\sharp]$ *is boundeded on* $L^{q(\cdot)}(\mathbb{Q}_p^n)$.

$(3) \Longrightarrow (4)$*: for any p-adic ball* $B_\gamma(x) \subset \mathbb{Q}_p^n$*, let* $y \in \mathbb{Q}_p^n$*, we have [7]*

$$M_p^\sharp(\chi_{B_\gamma(x)})(y) = \frac{2(p-1)}{p^2}. \tag{3.2}$$

Using (3) and (3.2)

$$\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \frac{p^2}{2(p-1)}\|[b, M_p^\sharp](\chi_{B_\gamma(x)})\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq C\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.$$

Then

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \leq C.$$

$(4) \Longrightarrow (1)$*: For any p adic ball* $B_\gamma(x) \subset \mathbb{Q}_p^n$*, let* $y \in \mathbb{Q}_p^n$*, we have [7]*

$$|b_{B_\gamma(x)}| \leq \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y). \tag{3.3}$$

$E = \{y \in B_\gamma(x) \,:\, b(y) \leq b_{B_\gamma(x)}\}$*, for any* $y \in E$*, we have:* $b(y) \leq b_{B_\gamma(x)} \leq \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)$.
Then

$$|b(y) - b_{B_\gamma(x)}| \leq |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|.$$

Using (4) and Lemmas 2.1, 2.5, we obtain

$$\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)} |b(y) - b_{B_\gamma(x)}|dy \leq \frac{2}{|B_\gamma(x)|_h}\int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|dy$$

$$\leq C\frac{\|b-\frac{p^2}{2(p-1)}M_p^{\sharp}(b\chi_{B_\gamma(x)})\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}\leq C, \tag{3.4}$$

which implies $b\in \mathrm{BMO}(\mathbb{Q}_p^n)$.

Next, we further proof $b^-\in L^\infty(\mathbb{Q}_p^n)$, *for any fixed* p-*adic ball* $B_\gamma(x)\subset\mathbb{Q}_p^n$. *If* $y\in B_\gamma(x)$, *by using (3.3) we obtain*

$$\frac{p^2}{2(p-1)}M_p^{\sharp}(b\chi_{B_\gamma(x)})(y)-b(y)\geq |b_{B_\gamma(x)}|-b^+(y)+b^-(y).$$

Thus

$$\begin{aligned}&\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)}|\frac{p^2}{2(p-1)}M_p^{\sharp}(b\chi_{B_\gamma(x)})(y)-b(y)|dy\\ &\geq\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)}\left|b_{B_\gamma(x)}\right|-b^+(y)+b^-(y)dy\\ &=\left|b_{B_\gamma(x)}\right|-\frac{1}{|B_\gamma(x)|_h}\left(\int_{B_\gamma(x)}b^+(y)dy-\int_{B_\gamma(x)}b^-(y)dy\right).\end{aligned}$$

Let $\gamma\to-\infty$ *and* $y\in B_\gamma(x)$ *we make use of (3.4) and Lebesgue differential theorem*

$$|b(y)|-b^+(y)+b^-(y)=2b^-(y)\leq C.$$

It deduce $b^-\in L^\infty(\mathbb{Q}_p^n)$. *Therefore, we finish the proof of Theorem 1.1.*

Proof of Theorem 1.2 *Since the implications* $(2)\Longrightarrow(3)$ *and* $(5)\Longrightarrow(4)$ *follow readily, and* $(2)\Longrightarrow(5)$ *is similar to* $(3)\Longrightarrow(4)$, *we only need to prove* $(1)\Longrightarrow(2),(3)\Longrightarrow(4),(4)\Longrightarrow(1)$.

$(1)\Longrightarrow(2)$. *Suppose* $b\in\Lambda_\beta(\mathbb{Q}_p^n)$, $b\geq 0$, *for any* $x\in\mathbb{Q}_p^n$. *It follows from (3.1) and Theorem 2 of [8] that* $[b,M_p^{\sharp}]$ *is bounded from* $L^{r,\lambda}(\mathbb{Q}_p^n)$ *to* $L^{q,\lambda}(\mathbb{Q}_p^n)$.

$(3)\Longrightarrow(4)$. *for any fixed* p-*adic ball* $B_\gamma(x)\subset\mathbb{Q}_p^n$, *using assertion (3), (3.2) and Lemma 2.6,*

$$\begin{aligned}\|b-\frac{p^2}{2(p-1)}M_p^{\sharp}(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}&\leq\frac{p^2}{2(p-1)}\|[b,M_p^{\sharp}](\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}\\ &\leq C|B_\gamma(x)|_h^{-\frac{\beta}{n}}\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}.\end{aligned}$$

Then

$$\frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}}\frac{\|b-\frac{p^2}{2(p-1)}M_p^{\sharp}(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}\leq C.$$

$(4)\Longrightarrow(1)$. *For any* p *adic ball* $B_\gamma(x)\subset\mathbb{Q}_p^n$, *let* $y\in\mathbb{Q}_p^n$, *we have [7]*

$$|b_{B_\gamma(x)}|\leq\frac{p^2}{2(p-1)}M_p^{\sharp}(b\chi_{B_\gamma(x)})(y). \tag{3.5}$$

$E = \{y \in B_\gamma(x) \ : \ b(y) \le b_{B_\gamma(x)}\}$, *for any* $y \in E$, *we have:* $b(y) \le b_{B_\gamma(x)} \le \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)$. *Then*

$$|b(y) - b_{B_\gamma(x)}| \le |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|. \tag{3.6}$$

By using Lemma 2.6 and the fact $\frac{1}{r} = \frac{1}{q} + \frac{\beta}{n-\lambda}$,

$$\begin{aligned}
&\frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} \left| b(y) - b\chi_{B_\gamma(x)} \right|^q dy \right)^{\frac{1}{q}} \\
&\le \frac{2}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|^q dy \right)^{\frac{1}{q}} \\
&= \frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n} + \frac{n-\lambda}{nq}}} \left(\frac{1}{|B_\gamma(x)|_h^{\frac{\lambda}{n}}} \int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|^q dy \right)^{\frac{1}{q}} \\
&\le C \frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \frac{\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \le C.
\end{aligned} \tag{3.7}$$

Using Remark 7(2), it deduce $b \in \Lambda_\beta(\mathbb{Q}_p^n)$

Next, we need to prove $b \ge 0$*, namely,* $b^- = 0$*. On the one hand, note that (3.3) of [20], we have*

$$\begin{aligned}
&\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |\frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)}) - b(y)| dy \\
&\ge |b_{B_\gamma(x)}| - \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} b^+(y) + \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} b^-(y) dy.
\end{aligned} \tag{3.8}$$

On the other hand, applying Hölder's inequality and (3.7)

$$\begin{aligned}
&\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |\frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)}) - b(y)| dy \\
&\le C \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} \chi_{B_\gamma(x)}(y) dy \right)^{1/q'} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |\frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)}) - b(y)|^q dy \right)^{1/q} \\
&\le C |B_\gamma(x)|_h^{\beta/n}.
\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$|b_{B_\gamma(x)}| - \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} b^+(y) + \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} b^-(y) dy \le C |B_\gamma(x)|_h^{\beta/n}. \tag{3.10}$$

Let $\gamma \to -\infty$ with $y \in B_\gamma(x)$, by virtue of p-adic version of Lebesgue differentiation theorem to (3.10), we obtain

$$|b(y)| - b^+(y) + b^-(y) = 2b^-(y) = 0,$$

Thus, we finish the proof of Theorem 1.2.

Remark 8 For the proof of Theorem 1.2, note that the estimation method is different from Theorem 2.1 in [29], which is briefly described as follows.

(i) (1) $\Rightarrow$ (2), the way can not only notice that b should be restricted to non-negative conditions, but also is relatively simple.

(ii) (2) $\Rightarrow$ (3), here, the norm estimate we deduce by boundedness is not the same as Theorem 2.1 of [29].

(iii) (4) $\Rightarrow$ (1), we avoid the use of Hölder's inequality in combination with the existing results, which is more helpful to understand the proof.

Proof of Theorem 1.3 *Since the implications* $(2) \Longrightarrow (3)$ *and* $(5) \Longrightarrow (4)$ *follow readily, and* $(2) \Longrightarrow (5)$ *is similar to* $(3) \Longrightarrow (4)$*, we only need to prove* $(1) \Longrightarrow (2), (3) \Longrightarrow (4), (4) \Longrightarrow (1)$.

$(1) \Longrightarrow (2)$. *Suppose $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$ and $b^- \in L^\infty(\mathbb{Q}_p^n)$; for any $x \in \mathbb{Q}_p^n$. It follows from (3.1) and Theorem 1.5 of [7] that $[b, M_p^\sharp]$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$.*

$(3) \Longrightarrow (4)$. *for any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, using assertion (3), (3.2), we obtain*

$$\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} \le \frac{p^2}{2(p-1)} \|[b, M_p^\sharp](\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} \le C\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}.$$

Then

$$\frac{\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \le C.$$

$(4) \Longrightarrow (1)$. *By virtue of Lemma 2.6 and (3.6), we have*

$$\begin{aligned}
&\left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|^q dy\right)^{\frac{1}{q}} \\
&= \frac{1}{|B_\gamma(x)|_h^{\frac{n-\lambda}{nq}}} \left(\frac{1}{|B_\gamma(x)|_h^{\frac{\lambda}{n}}} \int_{B_\gamma(x)} |b(y) - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})(y)|^q dy\right)^{\frac{1}{q}} \\
&\le C \frac{\|b - \frac{p^2}{2(p-1)} M_p^\sharp(b\chi_{B_\gamma(x)})\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}} \le C.
\end{aligned}$$

It follows from Theorem 1.4 of [7] that $b \in \mathrm{BMO}(\mathbb{Q}_p^n)$ and $b^- \in L^\infty(\mathbb{Q}_p^n)$. Thus we obtain Theorem 1.3.

Proof of Theorem 1.4 *By using the similar way of the proof of Theorem 1.5, Theorem 1.4 can be proven , hence, we obmit the proof.*

Proof of Theorem 1.5 *On the one hand, by Lemmas 2.2,2.4, we can get the result, of course the proof can be found in Theorem 2.2 of [29].*

On the other hand, if $M^b_{\alpha,p}$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$, for all fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$ and for any $y \in B_\gamma(x)$, we have

$$\begin{aligned}
&\frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}}\left(\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)}|b(y)-b_{B_\gamma(x)}|^q dy\right)^{\frac{1}{q}}\\
&\le \frac{1}{|B_\gamma(x)|_h^{\frac{\alpha+\beta}{n}}}\left(\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)}\left(\frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}}\int_{B_\gamma(x)}|b(y)-b(z)|\chi_{B_\gamma(x)}(z)dz\right)^q dy\right)^{\frac{1}{q}}\\
&\le \frac{1}{|B_\gamma(x)|_h^{\frac{\alpha+\beta}{n}}}\left(\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)}|M^b_{\alpha,p}(\chi_{B_\gamma(x)})(y)|^q dy\right)^{\frac{1}{q}}\\
&= \frac{1}{|B_\gamma(x)|_h^{\frac{\alpha+\beta}{n}}}\left(\left(\frac{|B_\gamma(x)|_h^{\frac{\lambda}{n}}}{|B_\gamma(x)|_h}\right)^{\frac{1}{q}}\left(\frac{1}{|B_\gamma(x)|_h^{\frac{\lambda}{n}}}\int_{B_\gamma(x)}|M^b_{\alpha,p}(\chi_{B_\gamma(x)})(y)|^q dy\right)^{\frac{1}{q}}\\
&\le C|B_\gamma(x)|_h^{-\frac{\alpha+\beta}{n}+\frac{\lambda}{nq}-\frac{1}{q}}\|\chi_{B_\gamma(x)}\|_{L^{r,\lambda}(\mathbb{Q}_p^n)} \le C.
\end{aligned}$$

The last step is obtained by the fact $\frac{1}{q} = \frac{1}{r} - \frac{\alpha+\beta}{n-\lambda}$. It follows from Remark 7(2) that $b \in \Lambda_\beta(\mathbb{Q}_p^n)$.

Remark 9 Note that $\Leftarrow$ is different from Theorem 2.2 in [29], we avoid the use of Hölder's inequality in combination with the existing results, which is more helpful to understand the proof.

For a fixed p-adic ball B_*, the fractional maximal function with respect to B_* of locally integrable function f is given by

$$M_{\alpha,B_*,p}(f)(x) = \sup_{\substack{\gamma\in\mathbb{Z}\\ B_\gamma(x)\subset B_*}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha}{n}}}\int_{B_\gamma(x)}|f(y)|dy,$$

where the supremum is taken over all the p-adic ball $B_\gamma(x)$ with $B_\gamma(x) \subset B_*$.

The following result play role in the proof of Theorem 1.7, for some details, we can see [19].

Lemma 3.1 Let b be a locally integral function on $\mathbb{Q}_p^n$ and $0 < \beta < 1$, $0 < \alpha < \alpha+\beta < n$, the following statements are equivalent.

(i) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$,

(ii) for all $1 \le q < \infty$, such that

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\gamma(x),p}(b)(y)|^q dy \right)^{\frac{1}{q}} < \infty, \tag{3.11}$$

(iii) for some s with $1 \le s < \infty$ such that (3.11) holds.

Proof of Theorem 1.6 *By using the similar way of the proof of Theorem 1.7, Theorem 1.6 can be proven , hence, we obmit the proof.*

Proof of Theorem 1.7 *Since $b \in \Lambda_\beta(\mathbb{Q}_p^n),\ b \ge 0$, for any $x \in \mathbb{Q}_p^n$, using Lemma 2.3, we obtain $[b, M_{\alpha,p}]$ is bounded from $L^{r,\lambda}(\mathbb{Q}_p^n)$ to $L^{q,\lambda}(\mathbb{Q}_p^n)$, the proof can also be found in Theorem 2.2 of [29].*

Next, note that $[b, M_{\alpha,p}] : L^{r,\lambda}(\mathbb{Q}_p^n) \to L^{q,\lambda}(\mathbb{Q}_p^n)$, for any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$ and for all $y \in B_\gamma(x)$, the following estimate is obtained in [19]

$$M_{\alpha,p}(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|_h^{\frac{\alpha}{n}} \quad , \quad M_{\alpha,p}(b\chi_{B_\gamma(x)})(y) = M_{\alpha,B_\gamma(x),p}(b)(y).$$

$$\begin{aligned}
&\frac{1}{|B_\gamma(x)|_h^{\frac{\beta}{n}}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) - |B_\gamma(x)|_h^{-\frac{\alpha}{n}} M_{\alpha,B_\gamma(x),p}(b)(y)|^q dy \right)^{\frac{1}{q}} \\
&= \frac{1}{|B_\gamma(x)|_h^{\frac{\alpha+\beta}{n}}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |b(y) M_{\alpha,p}(\chi_{B_\gamma(x)})(y) - M_{\alpha,p}(b\chi_{B_\gamma(x)})(y)|^q dy \right)^{\frac{1}{q}} \\
&= \frac{1}{|B_\gamma(x)|_h^{\frac{\alpha+\beta}{n}}} \left(\frac{|B_\gamma(x)|_h^{\frac{\lambda}{n}}}{|B_\gamma(x)|_h})^{\frac{1}{q}} (\frac{1}{|B_\gamma(x)|_h^{\frac{\lambda}{n}}} \int_{B_\gamma(x)} |[b, M_{\alpha,p}](\chi_{B_\gamma(x)})|^q dy \right)^{\frac{1}{q}} \\
&\le |B_\gamma(x)|_h^{-\frac{\alpha+\beta}{n} + \frac{\lambda}{nq} - \frac{1}{q}} \|\chi_{B_\gamma(x)}\|_{L^{r,\lambda}(\mathbb{Q}_p^n)} \le C.
\end{aligned}$$

The last step is obtained by the fact $\frac{1}{q} = \frac{1}{r} - \frac{\alpha+\beta}{n-\lambda}$. Using lemma 3.1, we get $b \in \Lambda_\beta(\mathbb{Q}_p^n),\ b \ge 0$.

Remark 10 Note that the estimation method is different from Theorem 2.3 in [29], where we avoid the use of Hölder's inequality in combination with the existing results, which is more helpful to understand the proof.

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Conflict of interest

The authors state that there is no conflict of interest.

Date availability statement

All data generated or analysed during this study are included in this published article.

References

[1] J. Bastero, M. Milman, F. Ruiz. Commutators for the maximal and sharp functions. *Proc. Amer. Math Soc*, 128(11):3329–3334, 2000.

[2] L.F. Chacón-Cortés, H. Rafeiro. Variable exponent lebesgue spaces and Hardy-littlewood maximal function on p-adic numbers. *p-Adic Numbers Ultrametric Anal. Appl.* 12(2):90–111, 2020.

[3] N.M. Chuong, D.V. Duong. Weighted Hardy-Littlewood operators and commutators on p-adic functional spaces. *p-Adic Numbers Ultrametric Anal. Appl.* 5(1):65–82, 2013.

[4] R. Coifman, R. Rochberg, G. Weiss. Factorization theorems for Hardy spaces in several variables. *Ann. Math.(2)* 103(3):611–635, 1976.

[5] B. Dragovich, N. Mišić. p-Adic hierarchical properties of the genetic code. *Biosystems*, 185:104017, 2019.

[6] D. Fan. Boundedness of the commutators of the maximal operator on Morrey spaces and their characterization of Lipschitz spaces. *Pure Math.* 13(12):3514-3524, 2023.

[7] Q. He and X. Li. Necessary and sufficient conditions for boundedness of commutators of maximal function on the p-adic vector spaces. *AIMS Mathematics* 8(6):14064–14085, 2023.

[8] Q. He, X. Li. Characterization of lipschitz spaces via commutators of maximal function on the p-adic vector space. *Journal of Mathematics*, 2022:Art. ID 7430272, 15 pages, 2022.

[9] S Janson. Mean oscillation and commutators of singular integral operators. *Ark. Mat.*, 16(1):263–270, 1978.

[10] D. Jordan, N. Mazza, S. Schroll, eds. *Modern trends in algebra and representation theory. Based on the LMS autumn algebra school 2020.* Cambridge University Press, Cambridge, 2023.

[11] Y.C. Kim. Carleson measures and the BMO space on the p-adic vector space. *Mathematische Nachrichten*, 282(9):1278–1304, 2009.

[12] K. Mahler. *Introduction to p-adic numbers and their functions.* Cambridge University Press, Cambridge, 1973.

[13] C.B. Morrey. On the solutions of quasi-linear elliptic partial differential equations. *Amer. Math. Soc.* 43(1):126–166, 1938.

[14] M. Paluszyński. Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss. *Indiana Univ. Math. J.* 44(1): 1–17, 1995.

[15] J.M. Rojas, Y. Zhu. A complexity chasm for solving sparse polynomial equations over p-adic fields. *ACM Communications in Computer Algebra*, 54(3):86–90, 2021.

[16] P. Schneider, U. Stuhler. The cohomology of p-adic symmetric spaces. *Invent. math*, 105(1):47–122, 1991.

[17] M. Taibleson. *Fourier analysis on local fields*, vol. 15. Princeton University Press, Princeton, 1975.

[18] V.S. Vladimirov, I.V.Volovich, E Zelenov. *p-adic analysis and mathematical physics*, vol 1. World Scientific, Singapore, 1994.

[19] J. Wu, P. Zhang. Characterization of Lipschitz functions via commutators of multilinear singular integral operators in variable exponent Lebesgue spaces. *Acta Math. Sin. (Engl. Ser)* 39(12):2465-2488, 2023.

[20] J. Wu, W. Zhao. Some estimates for commutators of the fractional maximal function on stratified Lie groups. *J. Inequal. Appl.* 2023: Art. ID 123, 17 pages, 2023.

[21] J. Wu and Y. Chang. Characterization of Lipschitz Spaces via Commutators of Fractional Maximal Function on the p-Adic Variable Exponent Lebesgue Spaces. *C. R. Math. Acad. Sci. Pairs* 362(1):177–194, 2024.

[22] P. Zhang. Characterization of Lipschitz spaces via commutators of the Hardy–Littlewood maximal function. *C. R. Math. Acad. Sci. Paris* 355(3):336–344, 2017.

[23] P. Zhang. Characterization of boundedness of some commutators of maximal functions in terms of Lipschitz spaces. *Anal. Math. Phys.* 9(3):1411–1427, 2019.

[24] P. Zhang and J. WU. Commutators of the fractional maximal functions. *Acta Math. Sinica (Chin. Ser.)* 52(6):1235–1238,2009.

[25] P. Zhang and J. Wu. Commutators for the maximal functions on Lebesgue spaces with variable exponent. *Math. Inequal. Appl.* 17(4):1375–1386, 2014.

[26] P. Zhang and J. Wu. Commutators of the fractional maximal function on variable exponent Lebesgue spaces. *Czechoslovak Math. J*, 64(1):183–197, 2014.

[27] P. Zhang, J. Wu, J. Sun. Commutators of some maximal functions with Lipschitz function on Orlicz spaces. *Mediterranean Journal of Mathematics*, 15(6):Art. ID 216, 13 pages, 2018.

[28] P. Zhang, Z. Si, J. Wu. Some notes on commutators of the fractional maximal function on variable Lebesgue spaces. *J. Inequal. Appl.* 2019:Art. ID 9, 17 pages, 2019.

[29] N, Sarfraz, M, B, Riaz, Q, A, Malik. Some new characterizations of boundedness of commutators of p-adic maximal-type functions on p-adic Morrey spaces in terms of Lipschitz spaces. *AIMS Mathematics*. 2024, 9(7): 19756-19770.